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Physics Classroom

Hermite Polynomials: A Heuristic Approach

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Abstract. In this note, I present different aspects of Hermite Polynomials derived from a heuristic point of view. Firstly, I will present a derivation of the generating function of the Hermite Polynomials given first few polynomials.

1. Generating Function

First four Hermite polynomials are given by

$$H_0(x) = 1 \tag{1}$$
$$H_1(x) = 2x \tag{2}$$
$$H_2(x) = 4x^2 - 2 \tag{3}$$
$$H_3(x) = 8x^3 - 12x \tag{4}$$

Here, we assume these polynomials as given, later in the note, I will present a simple method to derive them.

1.1. Recurrence relation by inspection—Notice that $H_2(x)$ can be written in terms of $H_1(x)$ and $H_0(x)$, namely,

$$H_2(x) = 2xH_1(x) - 2H_0(x) \tag{5}$$

Now let us attempt to write $H_3(x)$ in terms of $H_2(x)$ and $H_1(x)$

$$H_3(x) = 2xH_2(x) - 4H_1(x) \tag{6}$$

Notice that the above two equations can be rewritten as follows:

$$H_2(x) = 2xH_1(x) - 2(1)H_0(x) \tag{7}$$
$$H_3(x) = 2xH_2(x) - 2(2)H_1(x) \tag{8}$$

From this you can easily recognize that the multiplier in the bracket of the second term in the right hand side is same as the subscript of the Hermite polynomial in the first term of the right hand side. We can generalize a recurrence relation, by inspection as

$$H_{n+1} = 2xH_n(x) - 2nH_{n-1}(x) \tag{9}$$

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1.2. Derivation of generating function—$H_n(x)$, $n = 0, 1, 2, \cdots$ is a sequence of polynomials for $-\infty < x < \infty$. The generating function for generating the sequence of these polynomials can be defined as

$$F(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \tag{10}$$

Since each member of the sequence of polynomials is defined over $-\infty < x < \infty$, dividing by $n!$ will ensure the convergence of the series on the right hand side of the above equation for all x. Here t is a real parameter.

To obtain the generating fuction $F(x,t)$, multiply each term of eq. (9) with $t^n/n!$ and sum over all n from 0 to ∞. We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x) = 2x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) - 2 \sum_{n=0}^{\infty} n \frac{t^n}{n!} H_{n-1}(x) \tag{11}$$

Observe that the first term of the right hand side of eq.(11) is $2xF(x,t)$ by definition, see eq.(10).

Consider the second term on the right hand side of the eq.(11),

$$\begin{aligned} 2\sum_{n=0}^{\infty} n \frac{t^n}{n!} H_{n-1}(x) &= 2\sum_{n=1}^{\infty} n \frac{t^n}{n!} H_{n-1}(x) && (12) \\ &= 2\sum_{m=0}^{\infty} (m+1) \frac{t^{m+1}}{(m+1)!} H_m(x) && (13) \\ &= 2t \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x) && (14) \\ &= 2tF(t,x) && (15) \end{aligned}$$

The sum in the first equation starts from $n = 1$ since $n = 0$ term in zero. We set $n-1 = m$ in the second equation. In the third equation $(m+1)$ in the numerator is cancelled with the same factor in $(m+1)!$ in the denominator, leaving $m!$ in the denominator. We get the result by taking t outside the sum.

Consider the sum in the left hand side of eq.(11)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x) &= \sum_{m=1}^{\infty} \frac{t^{m-1}}{(m-1)!} H_m(x) && (16) \\ &= \sum_{m=1}^{\infty} m \frac{t^{m-1}}{m!} H_m(x) && (17) \\ &= \sum_{m=0}^{\infty} m \frac{t^{m-1}}{m!} H_m(x) && (18) \\ &= \frac{\partial}{\partial t} \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x) && (19) \\ &= \frac{\partial F(t,x)}{\partial t} && (20) \end{aligned}$$

In the first equation, we set $n+1 = m$. We get second equation by multiplying and dividing by m. We extend the sum from $m = 0$ in third equation since that term is anyway zero. We can write the summand in third equation as a derivative with respect to t as you can see in fourth equation. The sum is the generating function as per definition, as can be seen in fifth equation.

Now eq.(11) can be recast as a first order differential equation for generating function, given by

$$\frac{\partial F(t,x)}{\partial t} = (2x-2t)F(t,x) \tag{21}$$

Solving the above differential equation, we get the generating function for Hermite polynomials as

$$F(t,x) = e^{2tx-t^2} \tag{22}$$

2. Rodriguez Formula

Given the generating fuction $F(t,x)$, we can expand it in Taylor series at $t=0$.

$$F(t,x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial t^n} F(t,x)|_{t=0} \tag{23}$$

Comparing eq.(23) and eq.(10) we can obtain Hermite polynomials by successive differentiation of generating function evaluated at $t=0$,

$$H_n(x) = \frac{\partial^n}{\partial t^n} F(t,x)|_{t=0} \tag{24}$$

Substituting the generating function from eq.(22) in eq.(24), we get

$$H_n(x) = \frac{\partial^n}{\partial t^n} e^{2tx-t^2}|_{t=0} \tag{25}$$

Add and subtract x^2 in the power of the exponential to complete the square, we get

$$H_n(x) = \frac{\partial^n}{\partial t^n} e^{2tx-t^2+x^2-x^2}|_{t=0} \tag{26}$$

This equation can be written as

$$H_n(x) = e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2}|_{t=0} \tag{27}$$

Notice that $\partial/\partial t$ of $e^{-(t-x)^2}$ is same as $-\partial/\partial x$ of the same function. Thus we have

$$H_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-(t-x)^2}|_{t=0} \tag{28}$$

In the above equation, since the derivative is with respect to x, we can substitute $t=0$ even before differentiating. We get by setting $t=0$,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{29}$$

Equation (29) is called the Rodriguez formula and can be used to obtain Hermite polynomials by successive differentiation.

3. Hermite Polynomials

In this section, I will present a heuristic method to obtain the first four Hermite polynomials by using linear algebraic techniques.[1]

Let us represent a polynomial of degree 3, namely $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ as a column vector given by

$$f = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{30}$$

As we know that $\frac{d}{dx}f(x) = a_1 + 2a_2x + 3a_3x^2$, we can write

$$\frac{d}{dx}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} \tag{31}$$

Differential operator d/dx has the following matrix representation.

$$\frac{d}{dx} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

Similarly, we have $\frac{d^2}{dx^2}f(x) = 2a_2 + 6a_3x$ and the second derivative is represented by a matrix

$$\frac{d^2}{dx^2} \to \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

Further, $x\frac{d}{dx}f(x) = a_1x + 2a_2x^2 + 3a_3x^3$ and the corresponding matrix is given by

$$x\frac{d}{dx} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{34}$$

Let us now obtain a matrix corresponding to a differential operator

$$\hat{L} = \frac{d^2}{dx^2} - 2x\frac{d}{dx} \tag{35}$$

and the matrix is

$$\hat{L} \to \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \tag{36}$$

The matrix corresponding this operator is upper triangular and eigenvalues can be read out. We can see for $n = 0, 1, 2, 3$

$$\lambda_n = -2n \tag{37}$$

Note that $(\hat{L} + 2n)y = 0$ is the differential equation for Hermite polynomials. That implies we can get Hermite polynomials by finding out the eigenvectors of the matrix corresponding to the operator $\hat{L}$.

Eigenvectors corresponding to $\lambda_n = -2n, n = 0, 1, 2, 3$ respectively are

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix} \tag{38}$$

Representing these eigenvectors as polynomials, we get

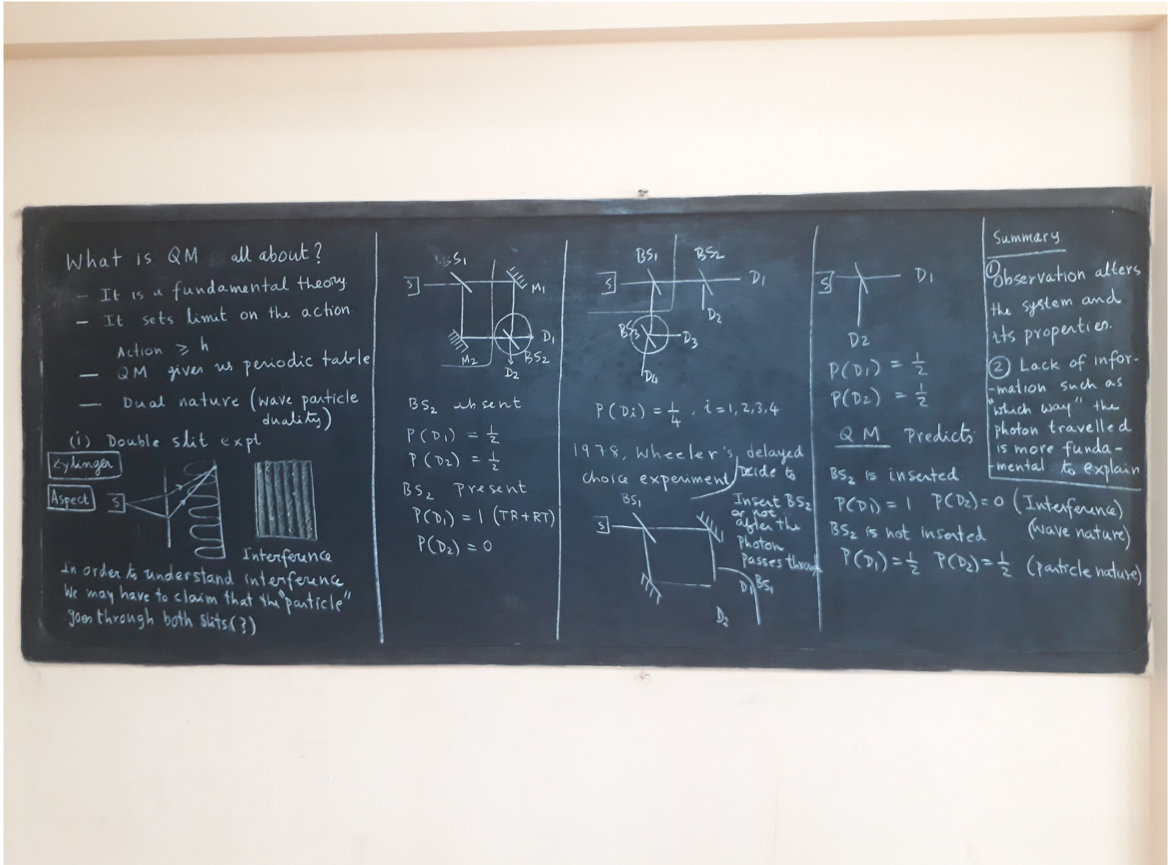
$$f_0(x) = 1, \; f_1(x) = x, \; f_2(x) = 1 - 2x, \; f_3(x) = 3x - 2x^3 \tag{39}$$

We get Hermite polynomials if they are normalized such that the highest power in n^{th} degree polynomial is $(2x)^n$. That is $H_n(x) = A_n f_n(x)$, where $A_0 = 1, \; A_1 = 2, \; A_2 = -2, \; A_3 = -4$. We get

$$H_0(x) = 1, \;\; H_1(x) = 2x, \;\; H_2(x) = 4x^2 - 2, \;\; H_3(x) = 8x^3 - 12x \tag{40}$$

Notes and References

[1] H. Beker, *Special polynomials by matrix algebra*, Am. J. Phys. **69** 812 (1998)


What is QM all about?
- It is a fundamental theory
- It sets limit on the action
Action ≥ h
- QM gives us periodic table
- Dual nature (wave particle duality)
(i) Double slit expt
Zeilinger
Aspect
Interference
In order to understand interference we may have to claim that the "particle" goes through both slits (?)
BS2 absent
P(D1) = 1/2
P(D2) = 1/2
BS2 Present
P(D1) = 1 (TR+RT)
P(D2) = 0
P(Di) = 1/4, i = 1,2,3,4
1978, Wheeler's delayed choice experiment
Decide to insert BS2 or not after the photon passes through BS1
P(D1) = 1/2
P(D2) = 1/2
QM Predicts
BS2 is inserted
P(D1) = 1 P(D2) = 0 (Interference) (wave nature)
BS2 is not inserted
P(D1) = 1/2 P(D2) = 1/2 (particle nature)
Summary
① Observation alters the system and its properties.
② Lack of information such as "which way" the photon travelled is more fundamental to explain


Heisenberg's 1925 work
Intensities are different for different spectral lines.
P(m→n) : Power radiated for a spectral line with frequency ωmn
Using 3 in the denominator
Use Kuhn-Thomas rule
Heisenberg